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LETTER TO THE EDITOR

Common algebraic structure for the Calogero–Sutherland models

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Abstract. We investigate a common algebraic structure for the rational and trigonometric Calogero–Sutherland models by using the exchange-operator formalism. We show that the set of Jack polynomials whose arguments are Dunkl-type operators provides an orthogonal basis for the rational case.

One-dimensional quantum integrable models with long-range interaction have attracted much interest, not only because of their physical significance, but also due to their beautiful mathematical structure. One such model is the Sutherland (trigonometric) model, which describes interacting particles on a circle [1]. The total momentum and Hamiltonian of the model are given by, respectively,

$$P_{\mathrm{s}} = \sum_{j=1}^{N} \frac{1}{\mathrm{i}} \frac{\partial}{\partial \theta_j} \qquad H_{\mathrm{s}} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial \theta_j^2} + \frac{1}{2} \sum_{j<k} \frac{\beta(\beta-1)}{\sin^2[(\theta_j - \theta_k)/2]} \tag{1}$$

where β is a real constant. Excited states for the Sutherland model are written in terms of the Jack symmetric polynomials.

Another example of long-range interaction is the Calogero (rational) model of interacting harmonic oscillators [2]:

$$H_{\mathrm{c}} = \frac{1}{2} \sum_{j=1}^{N} \left(-\frac{\partial^2}{\partial x_j^2} + x_j^2 \right) + \sum_{j<k} \frac{\beta(\beta-1)}{(x_j - x_k)^2} \,. \tag{2}$$

Excited states for this model are of the form $\psi(x) = \varphi(x)\psi_0^{(\mathrm{c})}(x)$, where $\varphi(x)$ is some symmetric polynomial and $\psi_0^{(\mathrm{c})}(x)$ is the ground-state wavefunction (see equation (17) below). The polynomial part $\varphi(x)$ can be obtained in principle [1, 2]; however, properties of orthogonal bases have not been clarified as much as in the case of the Sutherland model.

Due to the integrability of the Calogero model, the operator H_{c} belongs to a family of commuting differential operators (*conserved quantities* in physical terminology). Ujino and Wadati explicitly constructed polynomials that diagonalize the first two of them [3]. They further obtained an operator representation for the eigenfunctions and showed that they diagonalize the first two conserved quantities [4]. Polychronakos studied some special cases of the wavefunctions [5] by using the exchange-operator formalism [6, 7].

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Orthogonal polynomials associated with the Calogero model were investigated recently by Baker and Forrester [8]. Their proof of orthogonality is based on the orthogonality of another set of polynomials, which they call *generalized Jacobi polynomials*. They obtained the orthogonality of the Calogero case via some limiting procedure. In this letter, using the exchange-operator formalism [6, 7], we shall show that the algebraic structure of the Calogero model coincides exactly with that of the Sutherland model. As a consequence, we present a new type of operator representation of a basis that diagonalizes all of the conserved quantities simultaneously, and proves the orthogonality without taking a limiting procedure.

We start by reviewing the method of calculating the excited states for the Sutherland model [1]. Let us rewrite the operators (1) in terms of the variables $x_j = \exp(\mathrm{i}\theta_j)$; then

$$P_{\mathrm{s}} = \sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j} \qquad H_{\mathrm{s}} = \sum_{j=1}^{N} \left(x_j \frac{\partial}{\partial x_j} \right)^2 - \beta(\beta-1) \sum_{j<k} \frac{2x_j x_k}{(x_j - x_k)^2} \,. \tag{3}$$

The ground-state wavefunction for the model is

$$\psi_0^{(\mathrm{s})}(x) = \prod_{j<k} |x_j - x_k|^{\beta} \prod_{j=1}^{N} x_j^{-\beta(N-1)/2} \,. \tag{4}$$

To obtain the excited states, it is convenient to make a gauge transformation on the momentum and Hamiltonian:

$$\tilde{P}_{\mathrm{s}} = (\psi_0^{(\mathrm{s})})^{-1} \circ P_{\mathrm{s}} \circ \psi_0^{(\mathrm{s})} = \sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j} \tag{5}$$

$$\begin{aligned} \tilde{H}_{\mathrm{s}} &= (\psi_0^{(\mathrm{s})})^{-1} \circ H_{\mathrm{s}} \circ \psi_0^{(\mathrm{s})} - \tfrac{1}{12}\beta^2 N(N^2-1) \\ &= \sum_{j=1}^{N} \left(x_j \frac{\partial}{\partial x_j} \right)^2 + \beta \sum_{j<k} \frac{x_j + x_k}{x_j - x_k} \left(x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k} \right) . \end{aligned} \tag{6}$$

A basis of joint eigenspace for $\tilde{P}_{\mathrm{s}}$ and $\tilde{H}_{\mathrm{s}}$ are known as the Jack polynomials [9, 10]. The Jack polynomials $J_\lambda(x)$, indexed by the partitions $\lambda = (\lambda_1, \ldots, \lambda_N)$ of length $\leqslant N$, are uniquely determined by the following properties:

(i) $J_\lambda(x) = m_\lambda + \sum_{\mu(<\lambda)} u_{\lambda\mu} m_\mu$,
(ii) $J_\lambda(x)$ are eigenfunctions of $\tilde{H}_{\mathrm{s}}$,

where m_λ are the monomial symmetric functions, and $\mu < \lambda$ is defined by the dominance ordering [9]. Instead of the second property, we may impose the orthogonality with respect to the scalar product,

$$(f, g)_{\mathrm{s}} = \oint f(x^{-1}) g(x) \psi_0^{(\mathrm{s})}(x^{-1}) \psi_0^{(\mathrm{s})}(x) \prod_{j=1}^{N} \frac{\mathrm{d}x_j}{2\pi \mathrm{i} x_j} \tag{7}$$

where the integration contour is the unit circle in the complex plane.

Integrability of the Sutherland model, i.e. existence of a family of commuting operators that includes the Sutherland Hamiltonian, can be proved by using the exchange-operator formalism [6]. We first introduce so-called *Dunkl operators* [11]:

$$D_j = \frac{\partial}{\partial x_j} - \beta \sum_{k(\neq j)} \frac{1}{x_j - x_k} (s_{jk} - 1) \qquad (j = 1, \ldots, N) \tag{8}$$

where s_{ij} are elements of the symmetric group S_N. An element s_{ij} acts on functions of $x_1, \ldots, x_N$ as an operator which permutes arguments x_i and x_j. These operators satisfy the following properties:

$$\begin{aligned}
&[D_i, D_j] = [x_i, x_j] = 0 \\
&s_{ij}x_j = x_i s_{ij} \qquad s_{ij}x_k = x_k s_{ij} \qquad (k \neq i, j) \\
&s_{ij}D_j = D_i s_{ij} \qquad s_{ij}D_k = D_k s_{ij} \qquad (k \neq i, j) \\
&[D_i, x_j] = \delta_{ij}\left(1 + \beta \sum_{k(\neq i)} s_{ik}\right) - (1 - \delta_{ij})\beta s_{ij}\,.
\end{aligned} \tag{9}$$

We denote the algebra generated by the elements x_j, D_j and s_{ij} as $\mathcal{A}_{\mathrm{s}}$. We then introduce an $\mathcal{A}_{\mathrm{s}}$-module $\mathcal{F}_{\mathrm{s}}$ (*Fock space*) generated by the vacuum vector $|0\rangle_{\mathrm{s}} = 1$. The elements D_j of $\mathcal{A}_{\mathrm{s}}$ annihilate the vacuum vector, and s_{ij} preserve $|0\rangle_{\mathrm{s}}$:

$$D_j|0\rangle_{\mathrm{s}} = 0 \qquad s_{ij}|0\rangle_{\mathrm{s}} = |0\rangle_{\mathrm{s}}\,. \tag{10}$$

Further we define *Cherednik operators* $\hat{D}_j$ [12, 13]:

$$\begin{aligned}
\hat{D}_j &= x_j D_j + \beta \sum_{k(<j)} s_{jk} \\
&= x_j \frac{\partial}{\partial x_j} - \beta \sum_{k(<j)} \frac{x_k}{x_j - x_k}(s_{jk} - 1) - \beta \sum_{k(>j)} \frac{x_j}{x_j - x_k}(s_{jk} - 1) + \beta(j-1)\,.
\end{aligned} \tag{11}$$

Since the operators $\hat{D}_j$ commute with each other, they are diagonalized simultaneously by a suitable choice of the bases of $\mathcal{F}_{\mathrm{s}}$ [13, 14]. We introduce *non-symmetric Jack polynomials* $\mathcal{J}_w^{\lambda}(x)$ with $w \in S_N$, characterized by the following properties [13, 14]:

(i) $\mathcal{J}_w^{\lambda}(x) = x_w^{\lambda} + \sum_{(\mu,w')<(\lambda,w)} C_{ww'}^{\lambda\mu} x_{w'}^{\mu}$,

(ii) $\mathcal{J}_w^{\lambda}(x)$ are joint eigenfunctions for the operators $\hat{D}_j$,

where we have used the notation $x_w^{\lambda} = x_{w(1)}^{\lambda_1} \cdots x_{w(N)}^{\lambda_N}$. The ordering $(\mu, w') < (\lambda, w)$ is defined as follows:

$$(\mu, w') < (\lambda, w) \quad \Longleftrightarrow \quad \left\{\begin{array}{ll} \text{(i)} & \mu < \lambda, \\ \text{(ii)} & \text{if } \mu = \lambda \text{ then the first non-vanishing} \\ & \text{difference } w(j) - w'(j) \text{ is positive.} \end{array}\right. \tag{12}$$

For the element w_0 of S_N such that $w_0(j) = N - j + 1$ ($j = 1, \ldots, N$), eigenvalues of $\hat{D}_j$ are given by

$$\hat{D}_j \mathcal{J}_{w_0}^{\lambda}(x) = \left\{\lambda_{N-j+1} + \beta(j-1)\right\} \mathcal{J}_{w_0}^{\lambda}(x)\,. \tag{13}$$

For other elements $w \in S_N$, eigenvalues of $\hat{D}_j$ are all obtained by permutating the components of the multiplet $\left\{\lambda_{N-j+1} + \beta(j-1)\right\}_{j=1,\ldots,N}$.

Using $\hat{D}_j$, we introduce the generating function of commuting operators [13]:

$$\hat{\Delta}_{\mathrm{s}}(u) = \prod_{j=1}^{N} (u + \hat{D}_j)\,. \tag{14}$$

If we expand $\hat{\Delta}_{\mathrm{s}}(u)$ as a polynomial in u, the coefficients $\hat{I}_j^{(\mathrm{s})}$ form a set of commuting operators. The transformed momentum (5) and Hamiltonian (6) of the Sutherland model are related to $\hat{I}_j^{(\mathrm{s})}$;

$$\begin{aligned}
&\mathrm{Res}\, \hat{I}_1^{(\mathrm{s})} = \tilde{P}_{\mathrm{s}} + \tfrac{1}{2}\beta N(N-1) \\
&\mathrm{Res}\left((\hat{I}_1^{(\mathrm{s})})^2 - 2\hat{I}_2^{(\mathrm{s})}\right) = \tilde{H}_{\mathrm{s}} + \beta(N-1)\tilde{P}_{\mathrm{s}} + \tfrac{1}{6}\beta^2 N(N-1)(2N-1)
\end{aligned} \tag{15}$$

where Res X means that the action of X is restricted to symmetric functions of the variables $x_1, \ldots, x_N$.

Since $\hat{D}_{\rm s}(u)$ is symmetric in $\hat{D}_j$, symmetric eigenfunctions are obtained by symmetrizing $\mathcal{J}_w^\lambda(x)$, i.e. the Jack polynomials $J_\lambda(x)$ are the eigenfunctions. Eigenvalues of $\hat{\Delta}_{\rm s}(u)$ are given by

$$\hat{\Delta}_{\rm s}(u) J_\lambda(x) = \prod_{j=1}^{N} \left\{ u + \lambda_{N-j+1} + \beta(j-1) \right\} J_\lambda(x) \,. \tag{16}$$

Since all the eigenvalues of $\hat{\Delta}_{\rm s}(u)$ are distinct and the operator $\hat{D}_{\rm s}(u)$ is self-adjoint with respect to the scalar product (7), the Jack polynomials $J_\lambda(x)$ form an orthogonal basis with respect to the scalar product (7).

We then proceed to the Calogero model. The ground state for the Calogero Hamiltonian (2) is

$$\psi_0^{(\rm c)}(x) = \prod_{j<k} |x_j - x_k|^\beta \prod_{j=1}^{N} \exp\left(-\frac{x_j^2}{2} \right). \tag{17}$$

As in the case of the Sutherland model, the Calogero Hamiltonian is also related to the Dunkl operators D_j [6, 7]. We perform a kind of gauge transformation on $H_{\rm c}$:

$$\begin{aligned} \tilde{H}_{\rm c} &= \prod_{j<k} |x_j - x_k|^{-\beta} \circ H_{\rm c} \circ \prod_{j<k} |x_j - x_k|^\beta \\ &= \frac{1}{2} \sum_{j=1}^{N} \left(-\frac{\partial^2}{\partial x_j^2} + x_j^2 \right) - \frac{\beta}{2} \sum_{j \neq k} \frac{1}{x_j - x_k} \left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right). \end{aligned} \tag{18}$$

We then define an analogue of creation and annihilation operators,

$$A_j^\dagger = \frac{1}{\sqrt{2}}(-D_j + x_j) \qquad A_j = \frac{1}{\sqrt{2}}(D_j + x_j) \,. \tag{19}$$

We denote by $\mathcal{A}_{\rm c}$ an algebra generated by A_j, $A_j^\dagger$ and s_{ij}. Since the commutation relations of these operators are the same as those of x_j and D_j, we can introduce an isomorphism of $\mathcal{A}_{\rm s}$ to $\mathcal{A}_{\rm c}$ as follows:

$$\rho(x_j) = A_j^\dagger \qquad \rho(D_j) = A_j \,. \tag{20}$$

We note that this kind of isomorphism has already been used in [4]. Here we extend it to the isomorphism of Fock spaces. The Fock space for $\mathcal{A}_{\rm c}$ is constructed in the same way as $\mathcal{A}_{\rm s}$; Fock space $\mathcal{F}_{\rm c}$ is defined as $\mathcal{F}_{\rm c} = \mathbb{C}[A_1^\dagger, \ldots, A_N^\dagger]|0\rangle_{\rm c}$ where the vacuum vector $|0\rangle_{\rm c} = \prod_{j=1}^{N} \exp(-x_j^2/2)$ is annihilated by $A_j^\dagger$, i.e. $A_j^\dagger |0\rangle_{\rm c} = 0$. We denote also by ρ the isomorphism of $\mathcal{F}_{\rm s}$ to $\mathcal{F}_{\rm c}$ such that

$$\rho(|0\rangle_{\rm s}) = |0\rangle_{\rm c} \qquad \rho(a|v\rangle) = \rho(a)\rho(|v\rangle) \tag{21}$$

for $a \in \mathcal{A}_{\rm s}$ and $|v\rangle \in \mathcal{F}_{\rm s}$.

Since the operators $\hat{D}_j$ commute with each other, we can construct commuting operators $\hat{h}_j$ acting on $\mathcal{F}_{\rm c}$ as

$$\hat{h}_j = \rho(\hat{D}_j) = A_j^\dagger A_j + \beta \sum_{k(<j)} s_{jk} \,. \tag{22}$$

The generating function of commuting operators that include $\tilde{H}_{\rm c}$ is constructed by using $\hat{h}_j$:

$$\hat{\Delta}_{\rm c}(u) = \rho\big(\hat{\Delta}_{\rm s}(u)\big) = \prod_{j=1}^{N} (u + \hat{h}_j) \,. \tag{23}$$

We then define $\hat{I}_j^{(c)}$ as coefficients of $\hat{\Delta}_c(u)$:

$$\hat{\Delta}_c(u) = \sum_{j=0}^{N} u^{N-j} \hat{I}_j^{(c)} . \tag{24}$$

The transformed Calogero Hamiltonian $\tilde{H}_c$ is obtained as $\tilde{H}_c = \mathrm{Res}\, \hat{I}_1^{(c)} + N/2$. Our aim is diagonalization of $\hat{\Delta}_c(u)$ on $\mathcal{F}_c$. Since the Jack polynomials $J_\lambda(x)$ $(\in \mathcal{F}_s)$ diagonalize $\hat{\Delta}_s(u)$, we conclude that the vectors,

$$\rho\,(J_\lambda(x)|0\rangle_s) = J_\lambda(A^\dagger)|0\rangle_c \in \mathcal{F}_c \tag{25}$$

diagonalize $\hat{\Delta}_c(u)$. The eigenvalues of $\hat{\Delta}_c(u)$ are the same as those of $\hat{\Delta}_s(u)$ and all of them are distinct.

We then introduce another scalar product,

$$\begin{aligned} \langle\langle f, g\rangle\rangle &= {}_c\langle 0|f(A_1,\ldots,A_N)g(A_1^\dagger,\ldots,A_N^\dagger)|0\rangle_c \\ &= {}_s\langle 0|f(D_1,\ldots,D_N)g(x_1,\ldots,x_N)|0\rangle_s \end{aligned} \tag{26}$$

for f and g homogeneous polynomials of the same degree. The operator $A_j^\dagger$ is the adjoint of A_j with respect to this scalar product. Hence $\hat{\Delta}_c$ is self-adjoint for (26). It follows that the Jack polynomials are pairwise orthogonal relative to the scalar product (26). We note that the second expression of (26) is equivalent to the pairing introduced in [8] (equation (6.4) of [8]).

It is well known that the Jack polynomials are pairwise orthogonal for two kinds of scalar products; one is (7) and the other is a combinatorial one [9]. The scalar product (26) is the third example. Recently, Polychronakos calculated the norms of the elementary symmetric polynomials with respect to this scalar product [5]. The norms for the general Jack polynomials with respect to (26) were evaluated by Baker and Forrester [8]. In our notation, their result is written as

$$\langle\langle J_\lambda, J_\mu\rangle\rangle = \delta_{\lambda\mu} J_\lambda(x_1 = \cdots = x_N = 1) \prod_{(i,j)\in\lambda} \{\lambda_i - j + 1 + \beta(\lambda_j' - i)\} \tag{27}$$

with $\lambda' = (\lambda_1', \lambda_2', \ldots)$ the conjugate partition to λ. (We remark that the normalization of the Jack polynomials used in [8, 10] is in a different form from ours.) On the other hand, Stanley obtained the following formula ([10], theorem 5.4):

$$J_\lambda(x_1 = \cdots = x_N = 1) = \prod_{(i,j)\in\lambda} \frac{j - 1 + \beta(N - i + 1)}{\lambda_i - j + \beta(\lambda_j' - i + 1)}. \tag{28}$$

Combining these results, we finally come to the expression

$$\langle\langle J_\lambda, J_\mu\rangle\rangle = \delta_{\lambda\mu} \prod_{(i,j)\in\lambda} \frac{\{j - 1 + \beta(N - i + 1)\}\{\lambda_i - j + 1 + \beta(\lambda_j' - i)\}}{\lambda_i - j + \beta(\lambda_j' - i + 1)} . \tag{29}$$

In conclusion, we have proved that the algebraic structure for the Calogero model coincides exactly with that of the Sutherland model by using the exchange-operator formalism. We have further proved that the vectors (25) $\in \mathcal{F}_c$ form an orthogonal basis for the Calogero model. We hope that our results provide a useful tool for gaining a deeper understanding of the Calogero model.

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Note added in proof. After submission of this letter, a paper [15] was brought to the author's attention. In [15], Ujino and Wadati proved that the vectors (25) diagonalize the first two conserved quantities. However, they have not proved the orthogonality. We remark that the Rodrigues-type formula of [4, 15] is a consequence of the formula in [16] and the isomorphism (21) of the Fock spaces as is suggested in [4]. The author acknowledges Dr Hideaki Ujino for informing us of his results and for helpful comments.

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